THE POINCARÉ CONJECTURE

INTRODUCTION

- Motivation of the Poincare conjecture
- Steps to prove the conjecture.

THE POINCARE CONJECTURE TURNS INTO THEOREM.

- The definition of the Ricci flow.
- The Ricci flow with surgery.

IMPACT OF THE PROOF OF THE CONJECTURE.

- References

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INTRODUCTION - MOTIVATION OF THE POINCARÉ CONJECTURE

- 1904 (France) Henri Poincaré is studying the stability of the Solar System.
- In this work he was led to the Topology.
- The Poincaré conjecture (P.C.) was the first one in considering the possibility of chaos in a deterministic system, in its work on planetary orbits.
INTRODUCTION - MOTIVATION OF THE POINCARÉ CONJECTURE

Poincaré observed easily that any compact simply connected two-dimensional manifolds are topologically the same as the 2-sphere.*

He conjectured that it is also true in the case of three-dimensional manifolds.

THE POINCARÉ CONJECTURE
1904 Poincaré Conjecture.

many failed attempts become in very valuable scientific contributions.


1970 Michael Freedman . 4-spheres. Field medal.
**Introduction - Steps to Prove the Conjecture**

**Thurston Conjecture.** Every compact orientable three-manifold has a canonical decomposition into pieces, which admits a canonical geometric structure from among the 8 maximal simply connected homogeneous Riemannian 3-geometries.

**1982 Ricci flow equation.** It plays a main role in the demonstration.

**2002 Gregory Perelman.** Hamilton’s disciple. He proved the P.C. New concept of entropy and the surgery of the Ricci flow.
1. **DEFINITION OF THE RICCI FLOW**

The Ricci Flow equation:

\[ \partial_t g_{ij} = -2Ric_{ij}(g) \]

\(g\) is the Riemannian metric tensor and \(R\), the Ricci curvature tensor.

Rewriting the \(R\) as function of the Laplacian:

\[ \partial_t g = \Delta g \]

(similar to the Fourier equation)

This PDE describe how the curvature disperses in time.*
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A Ricci flow: one-parameter family of metrics $g(t)$, parameterized by $t$ in a non-degenerate interval $I$, on a smooth manifold $M$.

The initial condition: $(M, g(t_0)=g_0)$ (It does not have to exist)

Let $g_0$ be an Einstein metric:

$$
\begin{align*}
Ric(g_0) &= \lambda g_0 \\
g(t) &= cg_0
\end{align*}
$$

$$
\Rightarrow Ric(g) = \lambda g_0 = \frac{\lambda}{c} g
$$

$\lambda, c > 0, const.$

Using that, one can construct the solution
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Let’s assume that the one-parameter family of metrics \( g(t)=u(t)\ g_0 \) is a solution:

\[ \begin{align*}
\partial_t g &= u'(t)g_0 \\
\partial_t g &= -2\text{Ric}(u(t)g_0) = -2\text{Ric}(g_0) = -2\lambda g_0
\end{align*} \]

It “explodes” for \( t=1/2\lambda \).

\[ u'(t) = -2\lambda \rightarrow u(t) = 1 - 2\lambda t \]

Notice \( g(t_0) = u(t_0)g_0 = g_0 \Rightarrow u(t_0) = 1 \)

**Solution type solution:** \( g(t) = (1-2\lambda t)g_0 \)

Singularities of the Ricci flow equation.
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2. THE RICCI FLOW WITH SURGERY

Hamilton introduces the process known as surgery

It allows going on with the implementation of the flow till its extinction in a topological manifold (which satisfies the hypothesis of the P.C)!
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G. PERELMAN

THURSTON GEOMETRIES

NEW ENTROPY

Every singularity of the Ricci flow perfectly classified

GOOD SURGERY

it is easy to continue implementing the flow
The continued implementation of the flow before and after each surgery is carried out by a scaling operation.

One must check in each step that the obtained metric $g$ is a solution of the flow equation.

**n-th scaling**

Whether higher but finite values of $n$ are taken, one will obtain $g_n = \lambda_n g_0$ "renormalization"
THE POINCARE CONJECTURE TURNS INTO THEOREM.

- Summing up,
  - $\text{Ric} = \lambda g$
  - Consecutive implementations of the Ricci flow
    the final metric corresponds to a 3-sphere which is homeomorphic to the initial manifold.
  - Hence, it fulfills the requisite of being a simply connected 3-manifold.

G. PERELLEMAN PROVED THE POINCARE CONJECTURE
IMPACT OF THE PROOF OF THE CONJECTURE.

- The Poincaré Conjecture $\rightarrow$ one of the Seven Problems of the millennium.
- The Clay Mathematics Institute gives to G. Perelman 1 million dollars because of its proof of the P.C.
- Perelman rejected the award and he becomes internationally known.

The Poincaré Conjecture and G. Perelman are famous both in the scientific frame and in the whole world!
REFERENCES

